JOURNAL OF APPROXIMATION THEORY 77, 113–122 (1994)

Convergence Rate for Trigonometric Interpolation of Non-smooth Functions

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Communicated by Ronald A. DeVore

Received April 15, 1991; accepted in revised form January 29, 1993

The convergence rate of trigonometric interpolation operators in L^p norm is given in terms of the best one-sided approximation. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let f be a continuous function on $[0, 2\pi]$ and $L_n(f)$ be the trigonometric Lagrange interpolation polynomial on equidistant nodes in $[0, 2\pi)$. A classical theorem of Marcinkiewicz and Zygmund [9, Vol. II, p. 30] shows that $L_n(f)$ converges of f in L^p, $1 \leqslant p < \infty$, and

$$\|L_n(f) - f\|_p \leqslant \text{const } E_n(f)_\infty \leqslant \text{const } \omega\left(f; \frac{1}{n}\right)_\infty,$$

where $E_n(f)_\infty$ is the error of the best approximation by trigonometric polynomials of degree n in uniform norm, and $\omega(f; \delta)_\infty$ is the modulus of continuity of f. Recently, similar results have been proved for the mean convergence of $(0, m_1, ..., m_q)$ interpolation by trigonometric polynomials in [8]. For interpolation we do not really need continuity of the underlying function f. The interpolatory polynomial and the L^p error are well-defined already for bounded measurable functions f on $[0, 2\pi]$. To get L^p

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0021-9045/94 $6.00

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convergence of the Lagrange interpolation it is sufficient to assume Riemann integrability of f, which can be found already in the book of Zygmund [9, Chap. 10.7]. The purpose of this note is to obtain the order of convergence of $L_n(f)$ and, more generally, $(0, m_1, ..., m_q)$ interpolation in L^p norm for bounded measurable functions f. Since the interpolating polynomials are based on the point values of f, it is unrealistic to expect that the order be given by either $E_n(f)_p$ or $\omega(f; 1/n)_p$. Our order of convergence is given in terms of the error of the best one-sided approximation or in terms of the τ-modulus. However, if f is a smooth function, then we can give the order in terms of $E_n(f^{(s)})_p$. A typical example of the results is

$$\|L_n f - f\|_p \leqslant \text{const}\, n^{-1} E_n(f')_p, \qquad 1 < p < \infty,$$

provided f is an absolutely continuous function and $f' \in L^p$. For Lagrange interpolation, the above question has been dealt with by V. H. Hristov (see, e.g., [2]). The case of Jackson polynomials, i.e., the simplest case of Hermite–Fejér conditions, was treated by V. A. Popov and J. Szabados [4]. Our method is simpler and applies to the general $(m_1, ..., m_q)$ cases.

In Section 2, we introduce notations and the known results that are required. The case of the L^p norms, $1 < p < \infty$, is presented in Section 3. Here the Marcinkiewicz–Zygmund inequalities as proved in [5, 8] are the essential tools for the proof. The case $p = 1$ is of somewhat different character and is discussed in Section 4.

2. Preliminaries

We need the following two classes of trigonometric polynomials: $\mathscr{T}_M$ and $\mathscr{T}_{M,\varepsilon}$ ($\varepsilon = 0$ or 1). A trigonometric polynomial T belongs to $\mathscr{T}_M$, if

$$T(\theta) = a_0 + \sum_{k=1}^{M} (a_k \cos k\theta + b_k \sin k\theta),$$

and T belongs to $\mathscr{T}_{M,\varepsilon}$ if

$$T(\theta) = a_0 \sum_{k=1}^{M-1} (a_k \cos k\theta + b_k \sin k\theta) + a_M \cos\left(M\theta + \frac{\varepsilon\pi}{2}\right), \qquad \varepsilon = 0 \text{ or } 1.$$

If $0 < m_1 < \cdots < m_q$ are distinct integers, let E_q and O_q denote the number of even and odd integers in $(m_1, m_2, ..., m_q)$, respectively. It is known [1] that the problem of interpolation by $T(\theta)$ on the nodes $2k\pi/n$, $0 \leq k \leq n-1$, is regular for the case of $(0, m_1, ..., m_q)$ interpolation only in the following situations:

(I) $n = 2m+1$, $q = 2r$, $E_q - O_q = 0$, $T \in \mathscr{T}_M$, $M = nr + m$,

(IIa) $n = 2m+1$, $q = 2r+1$, $E_q - O_q = 1$, $T \in \mathscr{T}_{M,0}$, $M = nr + n$,

(IIb) $n = 2m+1$, $q = 2r+1$, $E_q - O_q = -1$, $T \in \mathscr{T}_{M,1}$, $M = nr + n$,

(III) $n = 2m$, $q = 2r$, $E_q - O_q = 0$, $T \in \mathscr{T}_{M,0}$, $M = nr + m$,

(IV) $n = 2m$, $q = 2r+1$, $E_q - O_q = -1$, $T \in \mathscr{T}_{M,1}$, $M = nr + n$.

Let B be the class of 2π-periodic functions that are bounded and measurable on $[0, 2\pi]$. For given n, q, and $f \in B$ we consider the interpolating operator $F_n f = F_{n,(m_1, ..., m_q)} f \in \mathscr{T}_M$ or $\mathscr{T}_{M,\varepsilon}$ (according to the conditions I–IV), which is uniquely defined by

$$F_n(f; \theta_{kn}) = f(\theta_{kn}), \qquad 0 \leq k \leq n-1,$$

$$F_n^{(j)}(f; \theta_{kn}) = 0, \qquad j = m_1, m_2, ..., m_q, \ 0 \leq k \leq n-1,$$

where $\theta_{kn} = 2k\pi/n$, $0 \leq k \leq n-1$. If $f^{(m_q)} \in B$, we also consider the interpolating polynomial $H_n f = H_{n,(m_1, ..., m_q)} f$, which is defined similarly as $F_n f$ but satisfies

$$H_n^{(j)}(f; \theta_{kn}) = f^{(j)}(\theta_{kn}), \qquad j = 0, m_1, ..., m_q, \ 0 \leq k \leq n-1.$$

In [5, 8], it is proved that $F_n f$ and $H_n f$ converge in L^p norm in the cases I, IIb, III, or IV, while the result in [3] pointed out that the convergence fails for the case IIa. Therefore, from now on we shall always assume that $(0, m_1, ..., m_q)$ satisfies one of the conditions I, IIb, III, or IV.

Let L^p, $1 \leq p < \infty$ be the space of 2π-periodic functions for which $\|f\|_p = (\int_0^{2\pi} |f(x)|^p \, dx)^{1/p} < \infty$, and C be the space of continuous 2π-periodic functions equipped with the uniform norm $\|f\|_\infty = \max_{0 \leq x \leq 2\pi} |f(x)|$. For $f \in L^p$, $1 \leq p < \infty$, or $f \in C$, we denote as usual the best approximation by trigonometric polynomials in $\mathscr{T}_n$ by $E_n(f)_p$, and the kth order L^p modulus by $\omega_k(f; t)_p$. The following theorem is proved in [5, 8].

Theorem A. *If $f \in C$, then*

$$\|F_n f - f\|_p \leq c_p \omega \left(f; \frac{1}{n}\right)_\infty, \qquad 1 \leq p < \infty,$$

and if $f^{(m_q)} \in C$, then

$$\|H_n f - f\|_p \leq c_p n^{-m_q} E_n(f^{(m_q)})_\infty, \qquad 1 \leq p < \infty.$$

In this theorem and throughout the rest of the paper the symbol c_p means a positive constant depending only on p and sometimes on the order of derivatives $\{m_1, ..., m_q\}$ or the order of τ-modulus, and c means an absolute constant. Their values may be different from line to line.

The proof of this theorem is based on the Marcinkiewicz–Zygmund inequality, which is also essential for our discussion in the case $1 < p < \infty$. We list it as

LEMMA 2.1. *Let $T \in \mathcal{T}_M$ or $\mathcal{T}_{M,\varepsilon}$ according to the case considered. Then* $(m_0 = 0)$

$$\|T\|_p \leqslant c_p \left\{ \sum_{j=0}^{q} \sum_{k=0}^{n-1} |T^{(m_j)}(\theta_{kn})|^p / n^{pm_j+1} \right\}^{1/p}, \qquad 1 < p < \infty.$$

We also need the following counterpart of Lemma 2.1 [8, Lemma 3; 9, Vol. II, p. 29].

LEMMA 2.2. *Let $1 \leqslant p < \infty$, and let $n > 0$, $r \geqslant 0$ be given integers. Then for any trigonometric polynomial T in $\mathcal{T}_{rn}$*

$$\left(\frac{1}{n} \sum_{k=0}^{n-1} |T(\theta_{kn})|^p \right)^{1/p} \leqslant c_r \|T\|_p.$$

To describe the order of convergence for $f \in B$ with respect to the L^p norm, we need the best one-sided approximation of f by means of trigonometric polynomials in $\mathcal{T}_n$, which is defined by

$$\tilde{E}_n(f)_p = \inf\{ \|P - Q\|_p : P, Q \in \mathcal{T}_n, Q(x) \leqslant f(x) \leqslant P(x), \forall x \}.$$

This quantity can be characterized by the averaged modulus of smoothness $\tau_k(f; \delta)_p$. Let $\omega_k(f, x, \delta)$ be the local modulus of smoothness of order k at x

$$\omega_k(f, x, \delta) = \sup\{ |\Delta_h^k f(t)| : t, t + kh \in [x - k\delta/2, x + k\delta/2] \},$$

where $\Delta_h^k f(t)$ is the kth difference with step h at t. Then

$$\tau_k(f; \delta)_p = \|\omega_k(f, \cdot, \delta)\|_p.$$

For the general properties of these quantities, we refer to the survey in [6]. Let W_s^p be the class of 2π-periodic functions defined by

$$W_s^p = \{ f \mid f^{(s-1)} \text{ is absolutely continuous and } f^{(s)} \in L^p \}.$$

We need the following

LEMMA 2.3. *If $f \in W_1^p$, then* [6, p. 184]

$$\tilde{E}_n(f)_p \leqslant \frac{2\pi}{n+1} E_n(f')_p, \tag{1}$$

and [6, p. 15]

$$\tau_k(f;\delta)_p \leqslant c\delta\omega_{k-1}(f';\delta)_p, \qquad k \geqslant 2, \delta > 0.$$

If $f \in L^p$, then [6, p. 169]

$$\tilde{E}_n(f)_p \leqslant c_k \tau_k\left(f;\frac{1}{n}\right)_p, \qquad n > k,$$

and [6, p. 14]

$$\omega_k(f;\delta)_p \leqslant \tau_k(f;\delta)_p, \qquad \delta > 0.$$

3. RESULTS FOR $p > 1$

THEOREM 3.1. *Let $n \geqslant 1$ and $f \in B$. Then*

$$\|F_n f - f\|_p \leqslant c_p\left[\tilde{E}_n(f)_p + \omega_{m_1}\left(f;\frac{1}{n}\right)_p\right], \qquad 1 < p < \infty. \tag{2}$$

Proof. Let $T_n \in \mathscr{T}_n$ be the best trigonometric polynomial approximation to f. Then

$$\|f - F_n f\|_p \leqslant \|f - T_n\|_p + \|T_n - F_n T_n\|_p + \|F_n(f - T_n)\|_p.$$

The first term on the right-hand side is just $E_n(f)_p$. According to a theorem in [7, Chap. 4.8.61],

$$\|T_n^{(r)}\|_p \leqslant c_p n^r \omega_r\left(f;\frac{1}{n}\right)_p,$$

where we note that the constant c_p depends also on r. Therefore, by Lemmas 2.1 and 2.2 the second term is bounded by

$$\begin{aligned}\|T_n - F_n T_n\|_p &\leqslant c_p\left(\sum_{j=1}^{q}\sum_{k=0}^{n-1} |T_n^{(m_j)}(\theta_{kn})|^p/n^{1+m_j p}\right)^{1/p}\\ &\leqslant c_p \sum_{j=1}^{q} \frac{1}{n^{m_j}} \|T_n^{(m_j)}\|_p\\ &\leqslant c_p \omega_{m_1}\left(f;\frac{1}{n}\right)_p,\end{aligned}$$

since $\omega_{r+1}(f;\delta)_p \leqslant 2\omega_r(f;\delta)_p$. To estimate the third term, let $\tilde{E}_n(f)_p = \|P_n - Q_n\|_p$, where $P_n, Q_n \in \mathscr{T}_n$ and $Q_n(x) \leqslant f(x) \leqslant P_n(x)$ for all x. Then by Lemma 2.1, Minkowski's inequality, and Lemma 2.2,

$$\begin{aligned}
\|F_n(f-T_n)\|_p &\leqslant c_p \left(\frac{1}{n}\sum_{k=0}^{n-1} |f(\theta_{kn}) - T_n(\theta_{kn})|^p\right)^{1/p} \\
&\leqslant c_p \left[\left(\frac{1}{n}\sum_{k=0}^{n-1} |P_n(\theta_{kn}) - Q_n(\theta_{kn})|^p\right)^{1/p} \right. \\
&\quad \left. + \left(\frac{1}{n}\sum_{k=0}^{n-1} |Q_n(\theta_{kn}) - T_n(\theta_{kn})|^p\right)^{1/p}\right] \\
&\leqslant c_p [\|P_n - Q_n\|_p + \|Q_n - T_n\|_p] \\
&\leqslant c_p [\tilde{E}_n(f)_p + \|Q_n - f\|_p + \|f - T_n\|_p] \\
&\leqslant c_p (2\tilde{E}_n(f)_p + E_n(f)_p).
\end{aligned}$$

Putting all these estimates together, and using Jackson's theorem

$$E_n(f)_p \leqslant c_p \omega_r\left(f;\frac{1}{n}\right)_p, \qquad r \geqslant 1,$$

completes the proof. ∎

In general, we cannot replace $\tilde{E}_n(f)_p$ by $\omega(f;1/n)_p$ in (2). However, the following corollary is true.

COROLLARY 3.2. *Let $f \in W_1^p$ and $m_1 > 1$. Then*

$$\|F_n f - f\|_p \leqslant c_p n^{-1} \omega_{m_1 - 1}\left(f';\frac{1}{n}\right)_p, \qquad 1 < p < \infty.$$

The proof of this corollary follows easily from Theorem 3.1 and Lemma 2.3. For the operator $H_n f$ we have

THEOREM 3.3. *Let $n \geqslant 1$, $q \geqslant 0$, and $f^{(s)} \in B$, $s = m_q$. Then*

$$\|H_n f - f\|_p \leqslant c_p n^{-s} \tilde{E}_n(f^{(s)})_p. \tag{3}$$

Proof. Since $H_n f$ preserves trigonometric polynomials in $\mathscr{T}_M$ or $\mathscr{T}_{M,\varepsilon}$ according to the case I–IV, we have

$$\|H_n f - f\|_p \leqslant \|f - T_n\|_p + \|H_n(f - T_n)\|_p,$$

where $T_n \in \mathscr{T}_n$ is the best trigonometric polynomial approximation to f. Let $Q_{n,j}$ and $P_{n,j}$ be the polynomials in $\mathscr{T}_n$, such that

$$\tilde{E}_n(f^{(m_j)})_p = \|P_{n,j} - Q_{n,j}\|_p \text{ and } Q_{n,j}(x) \leqslant f^{(m_j)}(x) \leqslant P_{n,j}(x) \qquad \text{for all } x,$$

$0 \leqslant j \leqslant q$. Then by Lemmas 2.1 and 2.2

$$\begin{aligned}
\|H_n(f - T_n)\|_p &\leqslant c_p \left(\sum_{j=0}^{q} \sum_{k=0}^{n-1} |f^{(m_j)}(\theta_{kn}) - T_n^{(m_j)}(\theta_{kn})|^p / n^{pm_j+1} \right)^{1/p} \\
&\leqslant c_p \left[\left(\sum_{j=0}^{q} \sum_{k=0}^{n-1} |P_{n,j}(\theta_{kn}) - Q_{n,j}(\theta_{kn})|^p / n^{pm_j+1} \right)^{1/p} \right. \\
&\quad \left. + \left(\sum_{j=0}^{q} \sum_{k=0}^{n-1} |Q_{n,j}(\theta_{kn}) - T_n^{(m_j)}(\theta_{kn})|^p / n^{pm_j+1} \right)^{1/p} \right] \\
&\leqslant c_p \left[\sum_{j=0}^{q} \|P_{n,j} - Q_{n,j}\|_p / n^{m_j} + \sum_{j=0}^{q} \|Q_{n,j} - T_n^{(m_j)}\|_p / n^{m_j} \right] \\
&\leqslant c_p \left[2 \sum_{j=0}^{q} \tilde{E}_n(f^{(m_j)})_p / n^{m_j} + \sum_{j=0}^{q} E_n(f^{(m_j)})_p / n^{m_j} \right],
\end{aligned}$$

where the last step follows from the inequality

$$\begin{aligned}
\|Q_{n,j} - T_n^{(m_j)}\|_p &\leqslant \|Q_{n,j} - f^{(m_j)}\|_p + \|f^{(m_j)} - T_n^{(m_j)}\|_p \\
&\leqslant \tilde{E}_n(f^{(m_j)})_p + c_p E_n(f^{(m_j)})_p.
\end{aligned}$$

Using (1) and the inequality $E_n(f)_p \leqslant c_p n^{-1} E_n(f')_p$ repeatedly, we get

$$\begin{aligned}
\|H_n(f - T_n)\|_p &\leqslant c_p \left[\tilde{E}_n(f^{(m_q)})_p + E_n(f^{(m_q)})_p \right] \Big/ n^{m_q}, \\
&\leqslant c_p n^{-s} \tilde{E}_n(f^{(s)})_p. \quad \blacksquare
\end{aligned}$$

COROLLARY 3.4. *Let* $n \geqslant 1$, $q \geqslant 0$, *and* $f \in W^p_{s+1}$, $s = m_q$. *Then*

$$\|H_n f - f\|_p \leqslant c_p n^{-s-1} E_n(f^{(s+1)})_p, \qquad 1 < p < \infty.$$

This corollary follows from (1) in Lemma 2.3. In particular, for Lagrange interpolation we have

COROLLARY 3.5. *Let* $n \geqslant 1$, $f \in W^p_1$. *Then*

$$\|L_n f - f\|_p \leqslant c_p n^{-1} E_{M-1}(f')_p, \qquad 1 < p < \infty.$$

If $f^{(s)}$ is a function of bounded variation ($f^{(s)} \in BV$), then it is known that [6, p. 10]

$$\tau(f^{(s)}; \delta)_p \leqslant c\delta^{1/p} V(f^{(s)}),$$

where $V(f^{(s)})$ is the variation of $f^{(s)}$ on $[0, 2\pi]$. Thus from Lemma 2.3 we get

COROLLARY 3.6. *Let* $n \geqslant 1$ *and* $f^{(s)} \in BV$, $s \geqslant 0$. *Then*

$$\|F_n f - f\|_p = O(n^{-\min\{m_1, s+1/p\}}), \qquad n \to \infty,$$

and for $s \geqslant m_q$

$$\|H_n f - f\|_p \leqslant c_p n^{-s-1/p} V(f^{(s)}).$$

4. RESULTS FOR $p = 1$

We now consider the L^1 norm estimate. The Marcinkiewicz–Zygmund inequality in the form of Lemma 2.1 is no longer available in this case, although there are some alternative forms for the Lagrange interpolation (see [9, Vol. II, p. 33]).

The L^1 norm estimate is related to the error of quadrature formulas. Let $\mathscr{L}_n(f)$ be the quadrature formula obtained by integrating $L_n(f; x)$ over $[0, 2\pi]$. Then the error

$$|e_n(f)| = \left|\int_0^{2\pi} f(x)\, dx - \mathscr{L}_n(f)\right| = \left|\int_0^{2\pi} (f(x) - L_n(f; x))\, dx\right|$$

can be bounded by $\tau(f; 1/n)_1$. For this and the related discussion, we refer to [6, Sect. 3.4, p. 60]. An estimate by $\tau(f; 1/n)_1$ for $\|f - L_n f\|_1$ would imply the estimate for $|e_n(f)|$. However, it is not likely that such an estimate for $\|f - L_n f\|_1$ is possible. In general, for $(0, m_1, ..., m_q)$ interpolation one can get L^1 norm estimate in the form of (2) or (3) only with an additional $\log n$ factor. However, for $L_n f$ such an estimate yields an order

$$\log n \tilde{E}_n(f)_1,$$

which is clearly too rough. Therefore, we do not pursue the estimate in this direction. Instead, we consider an alternative of $L_n f$, namely

$$L_{n,\nu} f = \sigma_{m,\nu}(L_n f), \qquad \nu = \left[\frac{m}{2}\right], \tag{4}$$

where $\sigma_{m,\nu} f$ are the de la Vallée Poussin means of the Fourier sum operator $S_m f$, i.e.,

$$\sigma_{m,\nu} f = \frac{1}{\nu+1} \sum_{k=m-\nu}^{m} S_k f$$

(see [9, p. 16]). Here we restrict ourselves to the case I with $q=r=0$, $n=2m+1=2M+1$. For $p>1$, the L^p norm estimate of $\sigma_{m,\nu} f - f$ follows readily from the results in Section 3. To estimate the L^1 norm, we need

LEMMA 4.1. *Let $T \in \mathscr{T}_M$. Then*

$$\|\sigma_{m,\nu} T\|_1 \leqslant c \left(\frac{1}{n} \sum_{k=0}^{n-1} |T(\theta_{kn})| \right).$$

Proof. Since $\nu = [m/2]$, it follows from [9, Vol. I, p. 115] that $\|\sigma_{m,\nu}\|_\infty$ is uniformly bounded. Thus we can prove this inequality by modifying the method in [9, Vol. II, p. 29]. We take a function g such that

$$\|\sigma_{m,\nu} T\|_1 = \int_0^{2\pi} (\sigma_{m,\nu} T)\, g\, dx, \qquad \|g\|_\infty = 1.$$

Since

$$\int_0^{2\pi} (S_k f)\, g\, dx = \int_0^{2\pi} (S_k f)(S_k g)\, dx = \int_0^{2\pi} f(S_k g)\, dx,$$

we have

$$\begin{aligned}
\|\sigma_{m,\nu} T\|_1 &= \int_0^{2\pi} (\sigma_{m,\nu} T)\, g\, dx = \int_0^{2\pi} T(\sigma_{m,\nu}\, g)\, dx \\
&= \frac{1}{n} \sum_{k=0}^{n-1} T(\theta_{kn})(\sigma_{m,\nu}\, g)(\theta_{kn}) \\
&\leqslant \|\sigma_{m,\nu}\|_\infty \cdot \|g\|_\infty \left(\frac{1}{n} \sum_{k=0}^{n-1} |T(\theta_{kn})| \right) \\
&\leqslant c \left(\frac{1}{n} \sum_{k=0}^{n-1} |T(\theta_{kn})| \right). \quad \blacksquare
\end{aligned}$$

THEOREM 4.2. *Let $f \in B$. Then for $\nu = [m/2]$*

$$\|L_{n,\nu} f - f\|_1 \leqslant c(E_{m-\nu}(f)_1 + \tilde{E}_m(f)_1).$$

Proof. Let $T_\nu \in \mathscr{T}_{m-\nu}$ satisfy $E_{m-\nu}(f)_1 = \|f - T_\nu\|_1$. Since it follows from the definition of $\sigma_{m,\nu} f$ that $\sigma_{m,\nu} T_\nu = T_\nu$, we also have $L_{n,\nu} T_\nu = \sigma_{m,\nu} L_n T_\nu = \sigma_{m,\nu} T_\nu = T_\nu$. Therefore

$$\|L_{n,\nu} f - f\|_1 \leqslant E_{m-\nu}(f)_1 + \|L_{n,\nu}(f - T_\nu)\|_1.$$

By (4) and Lemma 4.1 we have

$$\begin{aligned} \|L_{n,\nu}(f - T_\nu)\|_1 &= \|\sigma_{m,\nu} L_n (f - T_\nu)\|_1 \\ &\leqslant c\left(\frac{1}{n}\sum_{k=0}^{n-1} |f(\theta_{kn}) - T_\nu(\theta_{kn})|\right). \end{aligned}$$

The rest of the proof follows exactly as in the proof of Theorem 3.1. ∎

COROLLARY 4.3. *Let $n \geqslant 1$ and $f \in BV$. Then*

$$\|L_{n,\nu} f - f\|_1 \leqslant c n^{-1} V(f).$$

It is not clear if one can use $\sigma_{m,\nu}$ for the general $(0, m_1, ..., m_q)$ interpolation.

ACKNOWLEDGMENTS

The authors thank Dr. E. Quak and a referee for their helpful comments.

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